Lecture 2

6.2* - The Natural Logarithm

In Calc I you learned that

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

as long as $n \neq -1$. But, what happens if $n = -1$?

Def: We define the natural logarithm by

$$\ln x =$$

By construction, $\ln x$ is an anti-derivative for $\frac{1}{x}$.

Note: $\ln x$ is the area under $y = \frac{1}{t}$ between $t=1$ & $t=x$ for $x \geq 1$, and negative of the area under $y = \frac{1}{t}$ between $t=x$ and $t=1$ for $x<1$.

Upon defining a new function, it's good to see what properties it has:

Properties of $\ln x$

1) $D(\ln x) =$

2) $R(\ln x) =$

3)

x	>1	$=1$	<1
$\ln x$			

4) $\frac{d}{dx}(\ln x)$

5) The graph of $y = \ln x$ is

6) The function $f(x) = \ln(x)$ is

7) There exists a unique number, e, such that

Algebraic Properties of ln

i) $\ln 1 = 0$ ii) $\ln(ab) = \ln a + \ln b$

iii) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ iv) $\ln a^r = r \ln a$

Ex: Expand $\ln \frac{x^2\sqrt{x^2+1}}{x^3}$

Ex: Combine into a single logarithm

$$\ln x + 2\ln(x+1) - \frac{1}{3}\ln(x-1)$$

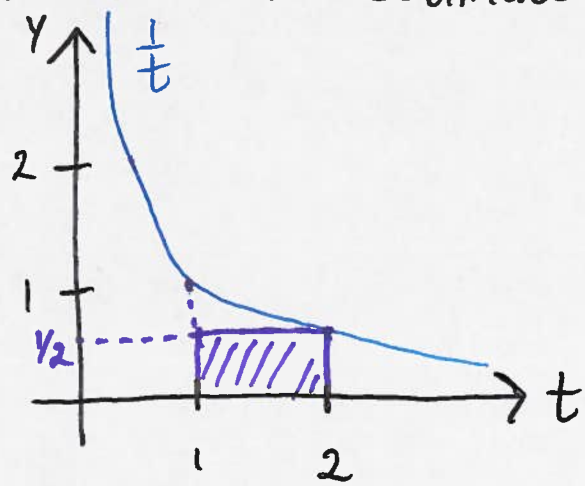
Ex: Evaluate $\int_1^{e^7} \frac{1}{t}\,dt$.

What are the values of:

$$\lim_{x\to\infty} \ln x \quad \& \quad \lim_{x\to 0} \ln x \ ?$$

Let's start with approximating $\ln 2$:

Using a Riemann sum with one rectangle and right endpoints, we get a lower estimate since $\frac{1}{t}$ is decreasing:



and so we get $\ln 2 > \frac{1}{2}(2-1) = \frac{1}{2}$.

Using properties of ln, we have then

$$\ln 2^n = n \ln 2 > \frac{n}{2}$$

So, if $x > 2^n$

and this implies

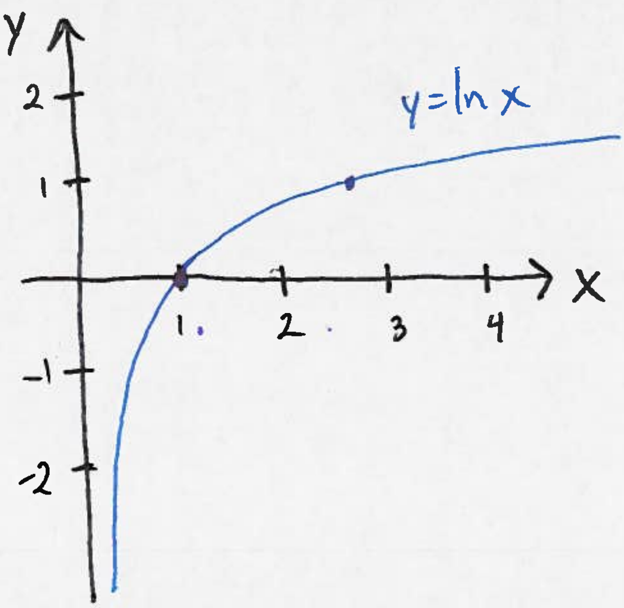
$$\lim_{x\to\infty} \ln x =$$

Likewise, $\ln \frac{1}{2^n} = -n \ln 2 < \frac{-n}{2}$. So, for $x < \frac{1}{2^n}$,

and this implies

$$\lim_{x \to 0} \ln x =$$

Using what we now know, we can sketch the graph of $y = \ln x$:



Ex: Compute $\lim_{x \to \infty} \left(\frac{x}{x^2 - 1} \right)$

A useful extension of ln (especially in differential equations) is obtained by composing it with the absolute value function:

$$\ln|x| = \begin{cases} \ln x, & x>0 \\ \ln(-x), & x<0 \end{cases}$$

It has the benefit of having a much larger domain than $\ln x$. Moreover, it is still an antiderivative of $\frac{1}{x}$, but now on the whole domain of it!

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \& \quad \int \frac{1}{x}\,dx = \ln|x| + C$$

Using the chain rule we get:

and by u-substitution:

Ex: Find $\frac{d}{dx}\left(\ln\left|\sqrt[3]{x-1}\right|\right)$

Ex: Compute $\int \frac{2x^3}{3-x^4}\,dx$

Logarithmic Differentiation

To differentiate $y=f(x)$, an often useful technique is logarithmic differentiation:

1) Take ln of both sides:

$$\ln y = \ln f(x)$$

2) Take the derivative w.r.t. x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\ln f(x)$$

3) Isolate $\frac{dy}{dx} = f'(x)$:

$$f'(x) = \frac{dy}{dx} = y\frac{d}{dx}\left(\ln f(x)\right) = f(x)\frac{d}{dx}\left(\ln f(x)\right)$$

Ex: Differentiate $y = \sqrt[7]{\frac{\cos^2 x}{(x^2+1)^2}}$